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On the Mathai–Quillen formalism of topological sigma models

Siye Wu¹

Department of Mathematics, Columbia University, New York, NY 10027, USA

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Abstract

We present a Mathai–Quillen interpretation of topological sigma models. The key to the construction is a natural connection in a suitable infinite-dimensional vector bundle over the space of maps from a Riemann surface (the world sheet) to an almost complex manifold (the target). We show that the covariant derivative of the section defined by the differential operator that appears in the equation for pseudo-holomorphic curves is precisely the linearization of the operator itself. We also discuss the Mathai–Quillen formalism of gauged topological sigma models.

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1. Introduction

The theory of pseudo-holomorphic curves has many successful applications to symplectic geometry since it was introduced by Gromov [1]. (For recent surveys, see [2] and references therein.) In the topological sigma model [3], one of the several topological field theories proposed by Witten, functional integrals are localized to the moduli space of pseudo-holomorphic curves in the (exact) semi-classical limit, the correlation functions are Donaldson-type invariants in Gromov’s theory and the space of quantum ground states is the Floer group. These phenomena, in this and other topological field theories, can be understood most naturally when the field theories are based on infinite-dimensional versions of the Mathai–Quillen construction [4]. (See [5] for a review.) For example, four-dimensional topological Yang–Mills theory [6], which is related to the works Donaldson and Floer, does

¹ E-mail: sw@shire.math.columbia.edu.

have a Mathai–Quillen interpretation [7]. In this paper, we present the topological sigma model and its gauged versions in the same spirit.

The paper is organized as follows. In Section 2, we review the Mathai–Quillen construction and its appearance in the infinite-dimensional setting of loop spaces. Its relation with the BRST algebra and the Lagrangian in supersymmetric quantum mechanics [8] will provide a guidance for the remaining sections. In Section 3, we consider the space of maps $\text{Map}(\Sigma, M)$ from a Riemann surface Σ to an almost complex manifold M and an infinite-dimensional vector bundle \mathcal{E}^{01} over $\text{Map}(\Sigma, M)$, whose fiber over a map u is the space of anti- J -linear sections of the bundle $u^*TM \otimes T^*\Sigma$ over Σ . We show that there is a natural connection ∇^{01} on \mathcal{E}^{01} . Moreover, the covariant derivative of the section $u \mapsto \bar{\partial}_J u$ along a tangent vector Φ of $\text{Map}(\Sigma, M)$ is the action of a first-order partial differential operator on Φ which linearize the section itself. This provides the Mathai–Quillen interpretation of the topological sigma model. Section 4 is about the gauged versions. We replace the maps to the target space in the previous model by sections of a fibration of Riemannian manifolds. We find out however that the connection of the infinite-dimensional vector bundle in this generalized setting preserves the linear metric in the fibers only when the parallel transport of the finite-dimensional fibered space generates isometries among the Riemannian fibers. In this special case, the topological sigma model is coupled to gauge fields in the usual sense.

2. Mathai–Quillen formalism and supersymmetric quantum mechanics

We first recall the basic notion of the Mathai–Quillen construction in finite-dimensional settings. Consider a vector bundle E of rank m associated to a principal bundle P over a compact base manifold of dimension n . Let $\{x^i\}$ be local coordinates on M and $\{\xi^a\}$, the linear coordinates on the fiber F . Choose a metric g_{ij} on M and a linear metric h_{ab} in the fiber to raise and lower indices. Given a connection ∇ on E (compatible with the metric h_{ab}), the Euler class $e(E)$ of the bundle E is the Pfaffian of the curvature 2-form R^a_b and can be written in terms of a fermionic integral. More generally, let ρ be a fermionic variable in F , then

$$u_\nabla(E) = \frac{1}{(2\pi)^m} e^{-|\xi|^2/2} \int d\rho e^{i\nabla\xi^a \rho_a + \rho_a R^{ab} \rho_b/2} \tag{2.1}$$

is a basic form on $P \times F$ and can be regarded as a representative of the Thom class on E [4]. For any section $s : M \rightarrow E$, the pull-back $e_{s,\nabla} = s^*u_\nabla(E)$ is given by the right-hand side of (2.1) after replacing ξ^a by s^a . The de Rham class of this m -form on M does not depend on the choice of the connection or the section, and is equal to the Euler class $e(E)$ of the bundle E . If $m = n$, we can integrate $e_{s,\nabla}$ over M ; this gives the Euler number $\chi(E)$. Introducing another fermionic variable χ in TM , then

$$\chi(E) = \frac{1}{(2\pi)^m} \int dx d\chi d\rho e^{-|s(x)|^2/2 + i\nabla_i s^a(x) \chi^i \rho_a + R^{ab}_{ij} \chi^i \chi^j \rho_a \rho_b/4}, \tag{2.2}$$

which resembles the partition function of a supersymmetric system. If $m < n$, we have to insert differential forms of appropriate degrees in the integrand in order to get nonzero numbers. Physically, this amounts to the calculation of the expectation value of an observable $O = O_{i_1 \dots i_{m-n}}(x) \chi^{i_1} \dots \chi^{i_{m-n}}$. In infinite-dimensional cases,² the Euler class so defined formally may depend of the choice of the section [7]. Moreover, the insertion of differential forms or observables is possible only when the difference $n - m$ is finite, or when the zero locus of the section is finite-dimensional.

Now consider a case in which the base manifold is the (infinite-dimensional) loop space $LM = \text{Map}(S^1, M)$ of a compact Riemannian manifold M with metric g_{ij} and the vector bundle is its tangent bundle $T(LM)$. A tangent vector at a loop $u(t)$ is a section of the pull-back bundle u^*TM over the circle S^1 . Choosing the functional-derivative operators $\delta/\delta u^i(t)$ as a basis of $T(LM)$, a tangent vector field on LM is locally

$$\Phi = \int_{S^1} dt \Phi^i(u, t) \frac{\delta}{\delta u^i(t)}, \quad (2.3)$$

or is simply denoted by $\Phi^i(u, t)$. LM is equipped with an induced metric

$$g \left(\frac{\delta}{\delta u^i(s)}, \frac{\delta}{\delta u^j(t)} \right) = g_{ij}(u(t)) \delta(s - t). \quad (2.4)$$

So the Christoffel symbols and the Riemann curvature of LM are equal to those of M up to factors of delta functions. For example, the covariant derivative of a vector field $\Phi^i(u, t)$ is

$$\nabla_{\delta/\delta u^i(s)} \Phi^k(u, t) = \frac{\delta \Phi^k(u, t)}{\delta u^i(s)} + \Gamma_{ij}^k(u(t)) \delta(s - t) \Phi^j(u, t). \quad (2.5)$$

For each Morse function W on M , there is a natural tangent vector field $\dot{u} + \text{grad } W$ on LM , whose components are $du^i(t)/dt + W^i(u(t))$. The covariant derivative of \dot{u} along a base vector is

$$\nabla_{\delta/\delta u^i(s)} \frac{du^k(t)}{dt} = \left[-\delta_i^k \frac{d}{ds} + \Gamma_{ij}^k \frac{du^j(t)}{dt} \right] \delta(s - t). \quad (2.6)$$

Hence for any tangent vector Φ of LM ,

$$\nabla_{\Phi} [\dot{u}^k(t) + W^k(u(t))] = D_t \Phi^k + \Phi^i W^k{}_{;i}, \quad (2.7)$$

where $D_t \Phi^k = \dot{\Phi}^k + \Gamma_{ij}^k \dot{u}^i \Phi^j$ is the covariant derivative given by the pull-back connection in u^*TM .

We now compare the Mathai–Quillen formalism of $T(LM)$ with supersymmetric quantum mechanics, of which the fundamental variables are a bosonic loop $u^i(t)$, two fermionic fields $\psi^i(t)$ and $\bar{\psi}^i(t)$ in the tangent space, and a (bosonic) multiplier field $B_i(t)$, with ghost numbers 0, 1, -1 , 0, respectively. The BRST algebra is (see for example [10,5])

² A systematic treatment of the notions of manifold, connection and curvature in infinite-dimensional settings can be found in [9].

$$\delta u^i = i\psi^i, \quad \delta\psi^i = 0, \tag{2.8}$$

$$\delta\bar{\psi}_i = B_i - i\Gamma_{ij}^k \bar{\psi}_k \psi^j, \tag{2.9}$$

$$\delta B_i = i\Gamma_{ij}^k B_k \psi^j - \frac{1}{2} R^j{}_{ikl} \bar{\psi}_j \psi^k \psi^l. \tag{2.10}$$

The Lagrangian is

$$\begin{aligned} \mathcal{L} &= \delta(\bar{\psi}_i(\dot{u}^i + W^{,i}) - \frac{1}{2}\bar{\psi}_i g^{ij} B_j) \\ &= -\frac{1}{2} B_i B^i + B_i(\dot{u}^i + W^{,i}) - i\bar{\psi}_i(D_t\psi^i + \psi^j W^{,i}{}_{;j}) + \frac{1}{4} R^{ij}{}_{kl} \bar{\psi}_i \bar{\psi}_j \psi^k \psi^l. \end{aligned} \tag{2.11}$$

After eliminating B_i using the equation of motion $B^i = \dot{u}^i + W^{,i}$, we have

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} g_{ij}(\dot{u}^i + W^{,i})(\dot{u}^j + W^{,j}) - i\bar{\psi}_i(D_t\psi^i + \psi^j W^{,i}{}_{;j}) \\ &\quad + \frac{1}{4} R^{ij}{}_{kl} \bar{\psi}_i \bar{\psi}_j \psi^k \psi^l. \end{aligned} \tag{2.12}$$

Both the BRST algebra and the Lagrangian are related to the Mathai–Quillen formalism of $T(LM)$. First, the noncovariant looking terms in (2.9) and (2.10) are determined by the term in (2.6) proportional to $du^j(t)/dt$, which is replaced by $\bar{\psi}^j$. $\delta^2\bar{\psi} = 0$ requires that the additional term in (2.10) is proportional to $\nabla^2\dot{u}$, i.e., the curvature of the infinite-dimensional bundle. That $\delta^2 B_i = 0$ is guaranteed by the (differential) Bianchi identity. Secondly, in light of (2.7), the action $\int_{S^1} dt \mathcal{L}$ agrees completely with the exponent in (2.2) with the section $s = \dot{u} + \text{grad } W$ and with $u(t)$, $\psi(t)$ and $\bar{\psi}(t)$ playing the role of x , χ , ρ , respectively. Hence the partition function

$$Z = \int \mathcal{D}u \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\int dt \mathcal{L}(u, \psi, \bar{\psi})} \tag{2.13}$$

is formally the Euler characteristic of LM regularized by the section $\dot{u} + \text{grad } W$.

Let F_u be the differential operator acting on Φ on the right-hand side of (2.7), i.e., $F_u\Phi = D_t\Phi + \nabla_\Phi(\text{grad } W)$. Mathematically, the above Mathai–Quillen interpretation is based on two facts. First $F_u\Phi = 0$ is precisely the linearization of the instanton equation

$$\dot{u} + \text{grad } W(u) = 0. \tag{2.14}$$

Secondly, due to the BRST algebra defined above, the same linearization also appears as the fermionic kinetic term in the Lagrangian (2.12). Furthermore, F_u determines the dimension of the space of solutions of (2.14). If u is a map from \mathbb{R} to M satisfying

$$\lim_{t \rightarrow -\infty} u(t) = y, \quad \lim_{t \rightarrow +\infty} u(t) = x, \tag{2.15}$$

where x and y are two (isolated) critical points of W with Morse indices $\text{ind}(x)$ and $\text{ind}(y)$ respectively, then F_u is a Fredholm operator whose index is (see for example [11])

$$\text{ind}(F_u) = \text{ind}(y) - \text{ind}(x). \tag{2.16}$$

Let $\mathcal{M}(y, x)$ be the space of solutions of (2.14) satisfying (2.15). For a generic metric g_{ij} (such that the gradient flow of W is of Morse–Smale type), F_u is onto and hence $\text{ind}(F_u)$ is equal to the dimension of $\mathcal{M}(y, x)$.

3. Mathai–Quillen interpretation of the topological sigma model

Topological sigma model is an analog of supersymmetric quantum mechanics in a more complicated situation. Instead of the loop space, we start with the space $\text{Map}(\Sigma, M)$ of maps from a Riemann surface Σ (with complex structure ε) to a symplectic manifold (M, ω) with a compatible almost complex structure J .³ Let g be the induced Riemannian metric on M . A natural generalization of the section $u \mapsto \dot{u}$ when $\Sigma = S^1$ is $u \mapsto du$, which is not a tangent vector of $\text{Map}(\Sigma, M)$. For each $u \in \text{Map}(\Sigma, M)$, du can be regarded as a section of the bundle $u^*TM \otimes T^*\Sigma$ over Σ . So $u \mapsto du$ is a section of a vector bundle $\mathcal{E} \rightarrow \text{Map}(\Sigma, M)$ whose fiber over u is $\mathcal{E}_u = \Gamma(u^*TM \otimes T^*\Sigma)$. Choosing local coordinates $\{\sigma^\alpha\}$ of Σ and $\{x^i\}$ of M , a local basis of $T(\text{Map}(\Sigma, M))$ is $\{\delta/\delta u^i(\sigma)\}$. A tangent field on $\text{Map}(\Sigma, M)$ has the form similar to (2.3):

$$\Phi = \int_{\Sigma} d^2\sigma \Phi^i(u, \sigma) \frac{\delta}{\delta u^i(\sigma)}. \tag{3.1}$$

A section of \mathcal{E} is

$$\Psi = \int_{\Sigma} d^2\sigma \Psi_\alpha^i(u, \sigma) \frac{\delta}{\delta u^i(\sigma)} \otimes d\sigma^\alpha, \tag{3.2}$$

or simply denoted by its components $\Psi_\alpha^i(u, \sigma)$. The bundle \mathcal{E} has a natural connection. Consider the evaluation map $\text{ev} : \text{Map}(\Sigma, M) \times \Sigma \rightarrow M$. Let π_1 and π_2 be the canonical projections of $\text{Map}(\Sigma, M) \times \Sigma$ onto $\text{Map}(\Sigma, M)$ and Σ , respectively. A section of \mathcal{E} can be canonically identified with one of $\text{ev}^*TM \otimes \pi_2^*T^*\Sigma$, i.e., $\Gamma(\mathcal{E}) \cong \Gamma(\text{ev}^*TM \otimes \pi_2^*T^*\Sigma)$. In fact, the bundle \mathcal{E} is the push-forward of $\text{ev}^*TM \otimes \pi_2^*T^*\Sigma$ via π_1 . The Levi-Civita connection on TM pulls back to ev^*TM . Choosing a metric $h_{\alpha\beta}$ on Σ compatible with the complex structure ε , $T\Sigma$ (hence $T^*\Sigma$) has a Levi-Civita connection, which pulls back to $\pi_2^*T^*\Sigma$. Thus we have a connection on $\text{ev}^*TM \otimes \pi_2^*T^*\Sigma$ by taking the tensor product. Finally, the connection on \mathcal{E} is obtained by restricting the covariant derivative to the tangent directions of $\text{Map}(\Sigma, M)$ in the base manifold. A simple calculation shows

$$\nabla_{\delta/\delta u^i(\tau)} \Psi_\alpha^k(u, \sigma) = \frac{\delta \Psi_\alpha^k(u, \sigma)}{\delta u^i(\tau)} + \Gamma_{ij}^k(u(\sigma)) \delta^{(2)}(\sigma - \tau) \Psi_\alpha^j(u, \sigma), \tag{3.3}$$

which is independent of the metric $h_{\alpha\beta}$ on Σ . After calculations similar to those to obtain (2.6) and (2.7), we find that the covariant derivative of the section $u \mapsto du$ along Φ is

$$\nabla_\Phi \partial_\alpha u^i(\sigma) = D_\alpha \Phi^i(\sigma), \tag{3.4}$$

where $D_\alpha \Phi^k = \partial_\alpha \Phi^k + \Gamma_{ij}^k \partial_\alpha u^i \Phi^j$ is given by the pull-back connection on u^*TM .

The problem with the bundle \mathcal{E} in the Mathai–Quillen construction is that the rank of \mathcal{E} is greater than the dimension of $\text{Map}(\Sigma, M)$ by an infinite amount. More precisely, the

³ The symplectic structure however is not important in the construction of the Lagrangian [3] or in the Mathai–Quillen interpretation.

linearization $d : \Gamma(u^*TM) \rightarrow \Gamma(u^*TM \otimes T^*\Sigma)$ of the section $u \mapsto du$ is not a Fredholm operator. This is resolved by restricting \mathcal{E}_u to its anti-holomorphic part $\mathcal{E}_u^{01} = \Gamma((u^*TM \otimes T^*\Sigma)^{01})$, i.e., the space of sections Ψ satisfying the “anti- J -linearity” constraint

$$\varepsilon_\alpha^\beta \Psi_\beta^i = -J^i_j \Psi_\alpha^j. \tag{3.5}$$

The sub-bundle \mathcal{E}^{01} of \mathcal{E} has a connection ∇^{01} obtained by projection, i.e.,

$$\nabla^{01} \Psi_\alpha^k(u, \sigma) = \frac{1}{2}(\nabla \Psi_\alpha^k(u, \sigma) + \varepsilon_\alpha^\beta J^k_j \nabla \Psi_\beta^j(u, \sigma)). \tag{3.6}$$

In \mathcal{E}^{01} , there is a natural section $u \mapsto \bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ \varepsilon)$, or $\bar{\partial}_\alpha u^i = \frac{1}{2}(\partial_\alpha u^i + \varepsilon_\alpha^\beta J^i_j \partial_\beta u^j)$. Solutions to the equation $\bar{\partial}_J u = 0$ are called pseudo-holomorphic (or J -holomorphic) curves in M [1]. The covariant derivative of the section $u \mapsto \bar{\partial}_J u$ along a tangent vector Φ at $u \in \text{Map}(\Sigma, M)$ is, taking into account the variation of the almost complex structure J ,

$$\nabla_\Phi^{01} \bar{\partial}_\alpha u^k = \frac{1}{2}(D_\alpha \Phi^k + \varepsilon_\alpha^\beta J^k_i D_\beta \Phi^i) + \frac{1}{4} J^k_{i;j} \Phi^j (\varepsilon_\alpha^\beta \partial_\beta u^i + J^i_l \partial_\alpha u^l). \tag{3.7}$$

In a coordinate-free language, for every $\Phi \in \Gamma(u^*TM)$,

$$\begin{aligned} \nabla_\Phi^{01}(\bar{\partial}_J) &= \frac{1}{2}(D\Phi + J \circ D\Phi \circ \varepsilon) + \frac{1}{4} D\Phi J \circ (du \circ \varepsilon + J \circ du) \\ &\in \Gamma((u^*TM \otimes T^*\Sigma)^{01}). \end{aligned} \tag{3.8}$$

If the target manifold M is Kähler, the complex structure J is covariantly constant, i.e., $J^k_{i;j} = 0$ or $\nabla_\Phi J = 0$. So the second term in (3.7) or (3.8) proportional to the covariant derivative of J drops out.

With this explicit expression, we can interpret the BRST algebra and the Lagrangian of the topological sigma model in [3].⁴ The fields consist of a (bosonic) map $u \in \text{Map}(\Sigma, M)$, two fermionic fields $\chi \in \Gamma(u^*TM)$, $\rho \in \Gamma((u^*TM \otimes T^*\Sigma)^{01})$, and a bosonic field H , a section in the same bundle as ρ . So the fields ρ_α^i and H_α^i obey the same “anti- J -linearity” constraint (3.5). At the classical level, there is a bosonic symmetry with charges $U = 0, 1, -1, 0$ on u, χ, ρ, H , respectively, corresponding to the grading of differential forms on the moduli space. The BRST supersymmetry is

$$\delta u^i = i \chi^i, \quad \delta \chi^i = 0, \tag{3.9}$$

$$\delta \rho_{\alpha i} = H_{\alpha i} - i(\Gamma_{ij}^k \delta_\alpha^\beta + \frac{1}{2} \varepsilon_\alpha^\beta J^k_{i;j}) \rho_{\beta k} \chi^j, \tag{3.10}$$

$$\begin{aligned} \delta H_{\alpha i} &= i(\Gamma_{ij}^k \delta_\alpha^\beta + \frac{1}{2} \varepsilon_\alpha^\beta J^k_{i;j}) H_{\beta k} \chi^j \\ &\quad - \frac{1}{4} (R^l_{ijk} + R^m_{njk} J^l_m J^n_i + J^l_{m;j} J^m_{i;k}) \rho_{\alpha l} \chi^j \chi^k. \end{aligned} \tag{3.11}$$

The terms proportional to χ^i in (3.10) and (3.11) correspond to those in (3.7) that are proportional to du^j . Since $\delta^2 \rho = 0$, the remaining terms in (3.11) give the curvature \mathcal{R} of the bundle \mathcal{E}^{01} . The Lagrangian can be chosen as $\mathcal{L} = \delta(\rho_i^\alpha (\partial_\alpha u^i - \frac{1}{4} H_\alpha^i))$. After eliminating H_α^i using the equation of motion $H_\alpha^i = \partial_\alpha u^i + \varepsilon_\alpha^\beta J^i_j \partial_\beta u^j$, we get

⁴ In [10], it was realized that this Lagrangian can be obtained by gauge fixing a topological action.

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} g_{ij} \partial_\alpha u^i \partial^\alpha u^j + \frac{1}{2} \varepsilon^{\alpha\beta} J_{ij} \partial_\alpha u^i \partial_\beta u^j - i \rho_i^\alpha (D_\alpha \chi^i + \frac{1}{2} \varepsilon_\alpha^\beta J^i_{j;k} \chi^k \partial_\beta u^j) \\ & + \frac{1}{8} (R^{ij}_{kl} + \frac{1}{2} J^i_{m;k} J^{jm}_{:l}) \rho_{\alpha i} \rho_j^\alpha \chi^k \chi^l. \end{aligned} \tag{3.12}$$

This is symbolically

$$\mathcal{L} \sim \frac{1}{2} |\bar{\partial}u|^2 - i\rho \nabla_\chi(\bar{\partial}) + \frac{1}{4} \mathcal{R} \chi \chi \rho \rho, \tag{3.13}$$

which agrees with the exponent of (2.2). Again, if M is a Kähler manifold, the BRST algebra and the Lagrangian are simplified due to the vanishing of the covariant derivatives of J , i.e., $J^k_{i;j} = 0$.

Similar to Section 2, Mathai–Quillen interpretation here is based on the result that the partial differential operator acting on Φ on the right-hand side of (3.7) or (3.8) is the linearization of the section $u \mapsto \bar{\partial}_J u$; this linearization again appears in the Lagrangian (3.12). Let \mathcal{M} be a connected component of the moduli space of pseudo-holomorphic curves $(\bar{\partial}_J)^{-1}(0)$. For a generic almost complex structure J , the linearization of $\bar{\partial}_J$ is onto; its index is equal to the dimension of \mathcal{M} . Applying Riemann–Roch theorem, we have [1]

$$\dim \mathcal{M} = (1 - g) \dim M + 2c_1(u^* T M). \tag{3.14}$$

To study the canonical formalism and Floer homology, take $\Sigma = \mathbb{R} \times S^1$ with coordinate $\sigma = t + is$. $T^* \Sigma$ has two global sections ds and dt satisfying $ds \circ \varepsilon = dt$ and $dt \circ \varepsilon = -ds$. A section Ψ of \mathcal{E}^{01} is of the form

$$\Psi = \frac{1}{2} \Psi_1 \otimes dt - \frac{1}{2} J \Psi_1 \otimes ds, \tag{3.15}$$

where Ψ_1 is a tangent vector field of $\text{Map}(\Sigma, M)$. In other words, \mathcal{E}^{01} and $T(\text{Map}(\Sigma, M))$ are isomorphic as bundles (though equipped with different connections). The section $\bar{\partial}_J u$, for example, corresponds to the tangent vector

$$(\bar{\partial}_J u)_i^j = \frac{\partial u^j}{\partial t} + J(u) \frac{\partial u^j}{\partial s}. \tag{3.16}$$

Naturally, (3.16) = 0 is the (nonlinear) Cauchy–Riemann equation for pseudo-holomorphic curves. One can introduce the analogue of the Morse potential in Section 2. Let $H : M \times S^1 \rightarrow \mathbb{R}$ be a (time-dependent) Hamiltonian function on the symplectic manifold M . Pulling back via the evaluation map, the gradient of H can be regarded as a tangent vector field of $\text{Map}(\Sigma, M)$, still denoted by $\text{grad } H$. According to the above discussion, the tangent vector

$$\frac{\partial u}{\partial t} + J \frac{\partial u}{\partial s} + \text{grad } H(u, s) \tag{3.17}$$

defines a section of \mathcal{E}^{01} . A direct calculation shows that the action of the covariant derivative ∇_Φ^{01} on (3.17) is

$$D_t \Phi + J D_s \Phi + \nabla_\Phi \text{grad } H + \frac{1}{2} \nabla_\Phi J \left(J \frac{\partial u}{\partial t} + \frac{\partial u}{\partial s} + J \text{grad } H \right). \tag{3.18}$$

In fact (3.17) = 0 is the gradient flow of a Morse function on LM whose critical points are periodic trajectories under the (time-dependent) Hamiltonian flow of H , under certain

topological assumptions on M and its symplectic form ω . If $u(s, t)$ does satisfies (3.17) = 0, then (3.18) reduces to

$$D_t \Phi + J D_s \Phi + \nabla_\phi \text{grad } H + \nabla_\phi J \circ \frac{\partial u}{\partial t}. \tag{3.19}$$

Here again, (3.19) is the linearization of (3.17). The index of the operator in (3.19) can be used to associate a grading on the set of periodic trajectories of the Hamiltonian flow [12] (see also [11]). The resulting Floer homology group is useful in solving the Arnold conjecture [13].

4. Gauged topological sigma models

In this section, we study the geometry of fibered Riemannian manifolds and discuss the corresponding Mathai–Quillen construction and the topological sigma model coupled to this geometric background, which includes an important special case of coupling to gauge fields.

Let $X \rightarrow \Sigma$ be a smooth fibration such that each fiber is diffeomorphic to a manifold M and is equipped with a (fiber-dependent) Riemannian metric g . Under a local trivialization, X can be described by the coordinates $\{\sigma^\alpha\}$ of Σ and $\{x^i\}$ of M , and $\{\partial_\alpha, \partial_i\}$ is a basis of the TX . The relative tangent bundle $T(X/\Sigma)$ of this fibration is a vector bundle over X whose fiber at each point is the tangent space to the fiber, i.e., spanned by $\{\partial_i\}$. The Levi-Civita connection on the fiber defines the parallel transport of vertical vectors along the vertical directions. We choose a splitting of TX into $T(X/\Sigma)$ and horizontal subspaces. Under an arbitrary local trivialization, ∂_α is not necessarily a horizontal vector. Let $f_\alpha^i \partial_i$ be its vertical component, then its horizontal component is $\tilde{\partial}_\alpha = \partial_\alpha - f_\alpha^i \partial_i$. This splitting defines a connection of $T(X/\Sigma)$, for it determines the parallel transport of points, and hence curves in the fibers along horizontal directions in X , and by differentiating, we know how to parallel transport vertical vectors along horizontal directions. Using the above coordinates, the covariant derivative is $\nabla_{\tilde{\partial}_\alpha} \partial_j = f_{\alpha,j}^i \partial_i$ or $\nabla_{\partial_\alpha} \partial_j = f_{\alpha,j}^i \partial_i$. So the Christoffel symbols $\Gamma_{\alpha j}^i = f_{\alpha,j}^i$. In a coordinate-free language, if H is the horizontal lift of a vector field on Σ and V is a vertical vector field on X , then $\nabla_H V = [H, V]$.

A section $u : \Sigma \rightarrow X$ is locally represented by $\sigma^\alpha \mapsto (\sigma^\alpha, u^i(\sigma))$. We define the covariant differential ∇u of u as the projection of $du = d\sigma^\alpha \otimes \partial_\alpha + \partial_\alpha u^i d\sigma^\alpha \otimes \partial_i$ onto the vertical directions, i.e., $\nabla u = (\partial_\alpha u^i + f_\alpha^i) d\sigma^\alpha \otimes \partial_i$. The tangent space of the space of sections $\Gamma(\Sigma, X)$ at u is $\Gamma(u^*T(X/\Sigma))$. Clearly, $u \mapsto \nabla u$ is a section of the bundle $\mathcal{E} \rightarrow \Gamma(\Sigma, X)$, with $\mathcal{E}_u = \Gamma(u^*T(X/\Sigma)) \otimes T^*\Sigma$. An arbitrary section of \mathcal{E} locally has the same form as (3.2) and can be identified with one of $ev^*T(X/\Sigma) \otimes \pi_2^*T^*\Sigma$ over $\Gamma(\Sigma, X) \times \Sigma$. Here $ev : \Gamma(\Sigma, X) \times \Sigma \rightarrow X$ is the evaluation map and π_2 is the projection of $\Gamma(\Sigma, X) \times \Sigma$ onto Σ . Taking the tensor product of the pull-back connections from $T(X/\Sigma)$ and $T^*\Sigma$, we get a connection on \mathcal{E} , which locally is still given by (3.3). Assume that Σ is a Riemann surface with complex structure ε and that there is an almost complex structure on each fiber of X , i.e., a section J of $\text{End}(T(X/\Sigma))$ such that $J^2 = -1$. We

restrict the bundle \mathcal{E} to its anti-holomorphic part, i.e., $\mathcal{E}_u^{01} = \Gamma((u^*T(X/\Sigma) \otimes T^*\Sigma)^{01})$, the space of sections satisfying (3.5). The sub-bundle \mathcal{E}^{01} has a connection ∇^{01} defined by projection (3.6). The natural section of \mathcal{E}^{01} is $u \mapsto \bar{\nabla}_J u = \frac{1}{2}(\nabla u + J \circ \nabla u \circ \varepsilon)$, or $\bar{\nabla}_\alpha u^i = \frac{1}{2}[(\partial_\alpha u^i + f_\alpha^i) + \varepsilon_\alpha^\beta J^i_j (\partial_\beta u^j + f_\beta^j)]$. Along any tangent vector $\Phi \in T_u \Gamma(\Sigma, X)$, the covariant derivative $\nabla_\Phi^{01}(\bar{\nabla}_J)$ is formally given by the same formula (3.7) or (3.8), but $D_\alpha \Phi^k = \partial_\alpha \Phi^k + (\Gamma_{ij}^k \partial_\alpha u^i + f_{\alpha;j}^k) \Phi^j$ is the pull-back connection on $u^*T(X/\Sigma)$.

Consider the topological sigma model coupled to this nondynamical fibration $X \rightarrow \Sigma$ as background, with $u \in \Gamma(\Sigma, X)$, $\chi \in \Gamma(u^*T(X/\Sigma))$ and $\rho, H \in \Gamma((u^*T(X/\Sigma) \otimes T^*\Sigma)^{01})$. They have the same statistics and the charge U as before. Moreover, the BRST algebra stays the same. (The terms in $\nabla_\Phi^{01}(\bar{\nabla}_J)$ proportional to $\partial_\alpha u^i$ do not change.) But the Lagrangian is replaced by

$$\mathcal{L} = \delta(\rho_i^\alpha (\nabla_\alpha u^i - \frac{1}{4} H_\alpha^i)). \tag{4.1}$$

After using the equation of motion $H_\alpha^i = \nabla_\alpha u^i + \varepsilon_\alpha^\beta J^i_j \nabla_\beta u^j$, \mathcal{L} has the same form as (3.12), except that $\partial_\alpha u^i$ is replaced by $\nabla_\alpha u^i$ and that D_α is the covariant derivative on $u^*T(X/\Sigma)$. So this sigma model coupled to the geometry of the fibration is topological in the sense that the Lagrangian is obtained by gauge-fixing the trivial action, and that the stationary phase approximation in the path integral is exact. However, the Mathai–Quillen interpretation works in the conventional sense only when the connection ∇^{01} in \mathcal{E}^{01} is metric-preserving. This would require that the connection ∇ in $T(X/\Sigma)$ is so, a statement not necessarily true. However, there is another natural connection on $T(X/\Sigma)$ that appeared in family index theorem [14,15] and topological gravity [16].

Recall that each fiber is equipped with a metric g_{ij} and that we have chosen horizontal subspaces at each point (locally characterized by f_α^i). Together with a metric $h_{\alpha\beta}$ on Σ , we can construct a metric

$$\tilde{g} = \begin{pmatrix} g_{ij} & g_{ij} f_\beta^j \\ f_\alpha^i g_{ij} & h_{\alpha\beta} + f_\alpha^i f_\beta^j g_{ij} \end{pmatrix} \tag{4.2}$$

on X such that the splitting of TX into vertical and horizontal subspaces is orthogonal under \tilde{g} . The projection of the Levi-Civita connection $\tilde{\nabla}$ on X onto the vertical directions defines a connection ∇' of $T(X/\Sigma)$. In terms of Christoffel symbols, this connection is given by

$$\Gamma'_{ij}{}^k = \tilde{\Gamma}_{ij}{}^k + \tilde{\Gamma}_{ij}{}^\beta f_\beta^k \tag{4.3}$$

and

$$\Gamma'_{\alpha j}{}^k = \tilde{\Gamma}_{\alpha j}{}^k + \tilde{\Gamma}_{\alpha j}{}^\beta f_\beta^k. \tag{4.4}$$

Along the vertical directions, $\tilde{\nabla}$ agrees with the Levi-Civita connection on the fiber, i.e., $\tilde{\Gamma}_{ij}{}^k = \Gamma_{ij}{}^k$. A more or less lengthy calculation shows that

$$\Gamma'_{\alpha j}{}^k = f_{\alpha;j}^k + \frac{1}{2} g^{ki} (L_{\tilde{\partial}_\alpha} g)_{ij}, \tag{4.5}$$

where

$$(L_{\tilde{\partial}_\alpha} g)_{ij} = g_{ij,\alpha} - (f_{\alpha,i}^k g_{kj} + f_{\alpha,j}^k g_{ik} + f_\alpha^k g_{ij,k}) \tag{4.6}$$

is the Lie derivative of g_{ij} with respect to the horizontal vector $\tilde{\partial}_\alpha$. (Here and after, the indices i, j, k, \dots are raised by the inverse g^{ij} of g_{ij} , not by $\tilde{g}^{ij} = g^{ij} + f_\alpha^i f_\beta^j h^{\alpha\beta}$.) Taking into account $\Gamma_{\alpha j}^i = f_{\alpha; j}^i$, (4.5) agrees with the coordinate-free expressions in [15].⁵

We now compare two connections ∇ and ∇' on $T(X/\Sigma)$. Both of them are independent of the metric $h_{\alpha\beta}$ on Σ and both are equal to the Levi-Civita connection of the fiber along vertical directions. The connection ∇' is metric preserving, but ∇ is not in general. In fact

$$\Gamma'_{\alpha j}{}^k - \Gamma_{\alpha j}{}^k = \frac{1}{2}g^{ki}(L_{\tilde{\partial}_\alpha}g)_{ij}. \tag{4.7}$$

So $\nabla' = \nabla$ if and only if $L_{\tilde{\partial}_\alpha}g = 0$, that is, when the parallel transport generates isometries among the fibers. This turns out to be a very important case. In general, the fibration $\pi : X \rightarrow \Sigma$ is an associated bundle of a principal $\text{Diff}(M)$ -bundle. Over each point $\sigma \in \Sigma$, the fiber of this principal bundle is $\text{Diff}(M, \pi^{-1}(\sigma))$, with the right action of $\text{Diff}(M)$ by composition. The connection of the principal bundle is defined by composing the maps in $\text{Diff}(M, \pi^{-1}(\sigma))$ with the parallel transport between the fibers. When $L_H g = 0$ for any horizontally lifted vector field H , the holonomy around any loop in Σ lies in the (finite-dimensional) compact Lie group G of isometries of M . So the structure group can be reduced to G and X is an associated bundle of a principal G -bundle P , i.e., $X = P \times_G M$.

If we start with a principal G -bundle P and assume that G acts on M preserving the metric g and the almost complex structure J , then the associated bundle $X = P \times_G M$ is a fibration of Riemannian manifolds with almost complex structures on the fibers. A connection on P , locally given by the gauge potential A_α^a on Σ , defines the horizontal subspaces in TX . Let $V_a, a = 1, \dots, \dim G$, be the Killing vector fields on M induced by the Lie algebra action, then under the induced local product structure of $X, f_\alpha^i = A_\alpha^a V_a^i$ and $\tilde{\partial}_\alpha = \partial_\alpha - A_\alpha^a V_a$. The Lie derivatives $L_{\tilde{\partial}_\alpha}g$ and $L_{A_\alpha^a V_a}g = A_\alpha^a L_{V_a}g$ are separately zero, hence $L_{\tilde{\partial}_\alpha}g = 0$. The Lagrangian is (3.12) with $\partial_\alpha u^i$ replaced by $\nabla_\alpha u^i = \partial_\alpha u^i + A_\alpha^a V_a^i$ and $D_\alpha \chi^k = \partial_\alpha \chi^k + (\Gamma_{ij}^k \partial_\alpha u^i + A_\alpha^a V_{a; j}^k)$. In this case, the connection ∇^{01} defined in the infinite-dimensional vector bundle \mathcal{E}^{01} is metric-preserving.

The BRST invariant observables are constructed from the G -equivariant differential forms on M [3]. (The latter is related to the differential forms on the symplectic quotient via the Kirwan map.) Each equivariant form on M can be extended to a form on X , pulled back to $\Gamma(\Sigma, X) \times \Sigma$, integrated along a homology cycle in Σ , and restricted to the moduli space \mathcal{M} in $\Gamma(\Sigma, X)$. It would be interesting to relate the intersection ring of \mathcal{M} to the cohomology ring of the symplectic quotient.

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⁵ In [14], it was shown that if there is a hermitian connection in a vector bundle over X , the induced connection in the (infinite-dimensional) push-forward bundle over Σ is not necessarily so. However it could be made hermitian by adding a term similar to the right-hand side of (4.7) below.

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